# Consequences of Indeterminate Form of Volume of Floating Balloons in Water†

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It is observed in critical analysis of completely submerged floating balloons in water, that under some feasible conditions, the volume of fluid filled in balloon takes indeterminate form i. e., $0/0$ in equations based upon Archimedes principle. These equations became feasible 1935 years after enunciation of the principle in 1685, when Newton defined $g$ in *The Principia*. If in this case definition of the principle is generalized i. e., upthrust is proportional to the weight of fluid displaced, then results are consistent. Thus, co-efficient of proportionality comes in picture, which accounts for shape of body, viscosity of medium, magnitude of medium etc. Stokes law takes in account the shape of body and viscosity of medium, and is experimentally confirmed , hence generalized form of the principle is justified, however in narrow range. Furthermore some specific experiments have been suggested to confirm effect of coefficient of proportionality. Such specific studies do not mean any comment or conclusion of the established status of the principle.

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## 1. Completely Submerged Floating Balloons Lead to Indeterminate Volume of Fluid Filled in Them

Archimedes principle was stated in 250 BC, and serves as a method for determination of densities of bodies. The modern mathematical equations became feasible after 1935 years when Newton published *The Principia* and defined acceleration due to gravity, $g$ in 1685. It must be noted that Archimedes principle [1, 2] was originally stated to determine the densities of bodies by measuring the volumes; and later on its applications were extended in rising, falling and floating bodies. These equations in case of completely submerged floating balloons/vessels, lead to inconsistent results in some cases described below.

Consider a balloon filled with medium of density $D_m$ completely submerged and floating in

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water of density $D_w$. The volume of sheath of balloon/vessel and additional mass is v and volume of medium filled inside the balloon is $V$ ( say wood, metal and gases). The mass of sheath and including additional mass attached to balloon is m. According to Archimedes principle the upthrust experienced by balloon is equal to weight of fluid displaced [1, 2]. The body displaces fluid equal to own volume.

$$VD_m g + mg = (V + v)D_w g \quad \text{or} \quad m = (V + v)D_w - VD_m. \tag{1}$$

From Eq. (1) different values of $V$, $D_w$, $D_m$ and $v$ can be written as

$$v = \frac{m - V(D_m - D_w)}{D_w}\,, \tag{2}$$

$$V = \frac{m - vD_w}{D_w - D_m}\,, \tag{3}$$

$$D_m = \frac{m + VD_w}{V + v}\,, \tag{4}$$

and

$$D_w = \frac{D_m(V + v) - m}{V}\,. \tag{5}$$

**1.1. Calculation of volumes $v$ and $V$.** Originally the principle was stated for measurement of densities of bodies by measuring their volumes. The volumes are given in Eqs. (2) and (3).

Now we can try to calculate the volume v of the sheath which also includes volume of mass $m$ (attached to it), when volume of the fluid filled in balloon is $V$ such that density of fluid filled inside is equal to that of water ($D_m = D_w$). Now substituting value of mass $m$ from Eq. (1) in Eq. (2)

$$v = \frac{(V + v)D_w - VD_w - V(D_w - D_w)}{D_w} = v, \tag{6}$$

which is true.

Now we can try to calculate the volume $V$ (wood, metal and gases) filled in floating balloon, the volume of the sheath of balloon and that of additional mass attached to balloon is v such that density of fluid filled inside balloon /vessel is equal to that of water ($D_m = D_w$). Obviously volume should turn out equal to $V$, which is actual value. For measurement of densities the volume of body ($V + v$) is required. Thus, the volume $V$ can be determined by substituting, mass from Eq. (1) in Eq. (3), we get

$$V = \frac{(V + v)D_w - D_w V - vD_w}{D_w - D_w} = \frac{vD_w - vD_w}{D_w - D_w} = \frac{0}{0}, \tag{7}$$

which is indeterminate form i. e., volume of medium filled in balloon (wood, metal and gases) becomes undefined but in actual experimental set up volume is $V$ consisting of metal, wood and gases.

Thus, RHS of Eq. (6) becomes devoid of units and dimensions which is not defined. Although division by zero is not permitted, yet it smoothly follows from equations based upon Archimedes' principle as per its definition. Also in this case numerator of the equation also becomes zero. The mathematical equations (hence definition of the principle) should be such that this situation should not arise. Further, $v$, $D_m$, $D_w$ cannot be calculated from Eqs. (3) – (5) due to Eq. (7). This experiment or perception has not been discussed even by Batchelor in standard treatise *An Introduction to*

*Fluid Dynamics* [3, Chap. I, § 1.4], A body "floating" in fluid at rest. This indicates the gravity and originality of the discussion.

L'Hospital's Rule is not applicable here, as it is exact equation, not a function or limit. No such method can give exact value of $V$. Thus, Archimedes' principle in original forms; in this particular case is not applicable. It can only be solved by generalizing 2265 years old Archimedes principle. Consider the following limit of the form $0/0$:

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x^3}.$$

The corresponding limit of derivatives is $\infty$:

$$\lim_{x\to 0} \frac{d(\sqrt{x+1}-1)/dx}{d(x^3)/dx} = \lim_{x\to 0} \frac{0.5(x+1)^{-1/2}}{3x^2} = \infty.$$

Hence, by L'Hospital's Rule,

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x^3} = \infty.$$

Thus, Archimedes' principles in original form in this particular case is not applicable. But it is true that when body is immersed in fluid, then its weight decreases as it experiences upthrust. So the generalization of the principle is only alternative. It may be understood in view of the fact that mathematical equations based upon Archimedes principle became feasible after 1935 years of enunciation of the principle when Newton published *The Principia* and defined acceleration due to gravity in 1685. Further, such results (indeterminate form of volume) have been reported after 326 years. Thus, implications of the deductions can be considered.

## 2. Generalization of Archimedes Principle

Under above conditions Archimedes principle becomes invalid mathematically. However practically in this case also the weight of balloon/body decreases in fluid. Thus, the principle has to be validated for all conditions; it can be so if its definition is generalized. Then equations in this particular case give logical results. The generalized form of the principle is "upthrust experienced by balloon is proportional to the weight of fluid displaced":

$$U_{gen} \alpha (V+v) D_w g \qquad \text{or} \qquad U_{gen} = f(V+v)D_w g, \tag{8}$$

where $f$ is coefficient of proportionality. The magnitude of $f$ is determined experimentally like all other coefficients of proportionality. If $f = 1$, then both original and generalized forms of the principle are same. Now upthrust can be slightly or infinitesimally less or more than weight of fluid displaced, depending upon value of $f$. Now Eqs. (1) and (2) become:

$$m = f(V+v)D_w - VD_m \quad \Rightarrow \quad V = \frac{m - fvD_w}{fD_w - D_m}. \tag{9}$$

Under the similar condition ($D_m = D_w$),

$$V = \frac{f(V+v)D_w - VD_w - fvD_w}{fD_w - D_m} = \frac{(f-1)D_w V}{(f-1)D_w} = V. \tag{10}$$

Now consistent results are obtained. Also now correct values of $v$, $D_m$ and $D_w$ are obtained. Unlike Eq. (6), in this case, no division by zero is involved, also numerator is non-zero, hence consistent and logical result is obtained. Correctly in this case the internal volume of balloon is $V$ which contains wood, metal and gases. From the theoretical derivation of Archimedes principle, it follows that $f$ has physical significance and takes in account those factors which are not taken in account by Archimedes principle.

### 3. Assumptions, Definitions and Hovering Kinematic Patterns

Consider in fluid of density $D_m$, block of height $H$ floats (under precisely static conditions and density of fluid is precisely uniform) such that upper surface of body is at depth $h$ in fluid. Also areas of upper and lower surfaces of body are regarded as precisely equal (say, $A$). The upthrust experienced by block $[1,2]$ is difference between thrusts at lower and upper surfaces i. e.,

$$\begin{aligned} u &= [\text{thrust at lower surface}] - [\text{thrust at upper surface}] \\ &= [p + D_m g(h+H)]A - (p + D_m gh)A = D_m HAg \end{aligned} \tag{11}$$

or

$$u = D_m Vg = [\text{weight of fluid displaced}]. \tag{12}$$

Now upthrust is equal to weight of fluid displaced if areas of upper and lower surfaces of the block are the same. But this derivation is not applicable for body of arbitrary shape, hence shape has a role to play even in theoretical derivation of Archimedes principle. The similar conclusions (the principle is not valid under all conditions and feasibility of its generalization) are drawn on the basis of Eq. (8). Further the experimental feasibility of deductions depend upon proposed experiments.

**3.1. Physical significance of coefficient of proportionality.** The coefficient of proportionality not have mathematical significance only but also have bearing on the experimental data. It can be justified in number of ways. According to Archimedes principle body floats if resultant weight is zero i. e.,

$$[\text{resultant weight}] = D_b Vg - D_m Vg = 0, \qquad D_b = D_m. \tag{13}$$

Thus, for body to float only densities of bodies and media are relevant, rest all others factors e. g. shape of body, magnitude, surface tension , viscosity of medium etc. are irrelevant.

In terms of the generalized form of Archimedes principle,

$$[\text{weight of body}] = [\text{upthrust exerted by medium}] \quad \Rightarrow \quad VD_b g = fVD_m g$$

or

$$D_b = fD_m. \tag{14}$$

Thus, when body floats then $f$ takes in account other factors (other than $D_m$ and $D_b$) e. g., shape of body, magnitude, surface tension, viscosity of medium, etc. Thus, theoretically generalized form of Archimedes principle is complete, as it takes all possible involved factors in account. The effect of these factors in such phenomena can be confirmed in specifically designed experiments.

**3.2. Experimental situation.** It must be noted that Archimedes principle was originally stated to determine the densities of bodies by measuring volumes; and later on its applications of the Archimedes principle were extended in rising, falling and floating bodies. But the explanation is purely qualitative only. It appears the principle will be more useful in the generalized form. The effects of shape and viscosity has been confirmed in experiments, in the region in which Stokes law is not valid. It can be tested in specific experiments over wide range of parameters.

***Stokes' Law.*** According to Stokes Law the bodies fall with constant velocity $c$ under certain conditions [4] i. e.,

$$c = \frac{2r^2}{9\eta}\left(1 - \frac{D_m}{D_b}\right) g D_b. \tag{15}$$

In 1910 Arnold [4] verified Eq. (15) in water with an accuracy of a few tenths of $1\ \%$ for sphere of rose metal of radii $0.002\ \mathrm{cm}$ i. e., $V = 33.524\cdot 10^{-9}\ \mathrm{cm}^3$. Stokes law is justified for extremely small bodies of particular shape i. e., spherical shape . The small spheres attain constant velocity due to viscous force of fluid. It implies that shape of body has been observed to influence the phenomena of falling bodies. Any how the shape of body is also a factor which may influence the results in motion of bodies; the observations are consistent with the generalised form of the principle.

***Archimedes Principle.*** Stokes law which confirms the effect of shape and viscosity of medium is applicable over a small range. But it establishes effect of shape of body and viscosity of medium upto some extent. It is physical significance of generalized form of Archimedes Principle. In general if we drop a steel body of mass $1\mathrm{kg}$ flat sheet ($1\ \mathrm{m} \times 1\ \mathrm{m}$ or $0.5\ \mathrm{m} \times 0.5\ \mathrm{m}$) or distorted shape and other body long pipe like (pointed ends) having mass $1\ \mathrm{kg}$. Then $1\ \mathrm{kg}$ body having shape long pipe like falls quickly than flat body of mass $1\ \mathrm{kg}$ ($1\ \mathrm{m} \times 1\ \mathrm{m}$ or $0.5\ \mathrm{m} \times 0.5\ \mathrm{m}$ or distorted shape). Both bodies of steel have same resultant weight (hence resultant acceleration) should fall equal distances in equal time, according to Archimedes principle. The flat body may be perforated and have different dimensions. But the distorted body or sheet falls slowly than long pipe like body, it is due to shape of body. Similar results can be obtained if bodies of different masses are studied.

The effect of shape may be specifically tested for floating balloons/bodies in more viscous fluids. The density of glycerine $1.26$ times that of water but the coefficient of viscosity of glycerine is $1058$ times that of water. If a body of density $1.26001\ \mathrm{g/cm}^3$ floats in glycerine, then coefficient of viscosity is slightly less than one can be discussed. The viscosity of fluids was studied in beginning of the 19-th century. This effect can be checked experimentally in specific experiments.

Thus, $f$ has not only mathematical significance but also physical significance that it takes in account those factors which are neglected in Archimedes principle e. g. shape of body, viscosity of medium, magnitude of medium etc. This discussion is only valid for completely submerged floating balloons under the given conditions. Some experiments regarding motion of bodies are

**Table 1.**
Consequences of indeterminate form of volume

| No. | Characteristic | Original form of Archimedes principle | Generalized form of Archimedes principle |
|---|---|---|---|
| 1 | Definition | $U = V D_m g$ | $U = f V D_m g$ |
| 2 | Volume under certain conditions | $V = 0/0$ | $V = V$($500\ \mathrm{cm}^3$, say) |
| 3 | Coefficient of proportionality | $f = 1$ | $f < 1$ or $f \geq 1$ |
| 4 | Status of parameters | Does not account for shape of body and viscosity of medium | Takes in account the shape of body and viscosity of medium |
| 5 | Specific experiments | Break down under certain conditions | Some specific experiments are suggested to determine effects of shape and viscosity of medium for completely submerged bodies |

mentioned only for completeness as they appear to support theoretical dependence of upthrust on shape of bodies etc. These effects can be confirmed in specific experiments, as these experiments and mathematical equations became feasible 1935 years of enunciation of the principle. The various deductions are shown in Table 1.

But this discussion has absolutely no effect on the experimentally confirmed data based upon Archimedes principle.

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